GENERATION OF TAYLOR VORTICES IN COUETTE FLOW

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Recently the study of mechanisms of generation of hydrodynamic turbulence has been the subject of numerous investigations (see [1] and the reviews [2, 3]). Until now, however, no answers have been found to a number of questions involving processes of transition from a laminar to a turbulent state. It should be mentioned that instability of laminar flow does not lead at once to turbulence: Other forms of equilibrium motion can develop consisting of a superposition of the initial flow and perturbations of finite amplitude. The flow between coaxial cylinders, Couette flow, which has been studied for many years, both theoretically and experimentally [4-6], can serve as such an example. It is known that upon an increase in the rotation rate of the inner cylinder Couette flow loses stability, which leads to the appearance of Taylor vortices. The present report is devoted to a detailed investigation of this transition. It includes both theoretical and experimental results.

In the course of the experiments it was established that the transition to Taylor vortices takes place in a mild way: With an increase in supercriticality the amplitude of the vortices varies by the Landau law [1]. The transitions between states with different numbers of vortices at a low supercriticality were also investigated and their stability limits were determined. In the theoretical part of the report the stability of steady-state solutions is investigated within the framework of the envelope equation describing weak spatial modulation of Taylor vortices.

1. As is known [6], stability loss of laminar Couette flow, the velocity of which is $V_\varphi = \frac{\Omega R_1^2}{R_2^2 - R_1^2}\left(\frac{R_2^2}{r} - r\right)$ for rotation of the inner cylinder (Ω is the rotation frequency and R_1 and R_2 are the radii of the inner and outer cylinders), sets in at some Reynolds number Re_* ($\mathrm{Re} = \Omega R_1^2/\nu$). This instability is aperiodic, and for perturbations which are periodic along the cylinder axis it is characterized by an increment γ_k having a maximum at some value of the wave number $k = k_0$. For low supercriticalities $\varepsilon = (\mathrm{Re} - \mathrm{Re}_*)/\mathrm{Re}_*$ the perturbations are unstable in a narrow interval near k_0. Their increment can be represented in the form

$$\gamma_k = \gamma_0 - \alpha(k - k_0)^2, \tag{1.1}$$

where $\gamma_0 \sim \varepsilon$; $\alpha > 0$, and the interval Δk of unstable wave numbers, as follows from this, is proportional to $\varepsilon^{1/2}$. Therefore, a narrow wave packet is excited at small excesses above the instability threshold. Thus, in the linear stage of instability a perturbation of the velocity $\mathbf{v}$ can be represented in the form

$$\mathbf{v} = \varphi(r)\, A(z, t)\, e^{ik_0 z} + \text{c.c.},$$

where $\varphi(r)e^{ik_0 z}$ is the eigenfunction of the linear problem with $k = k_0$; $A(z, t)$ is the amplitude of the perturbations, which varies slowly along z. As follows from the form of the increment (1.1), in the linear stage the amplitude A obeys the equation

$$\partial A/\partial t = \gamma_0 A + \alpha \partial^2 A/\partial z^2.$$

Stabilization of the instability is provided for by the nonlinear terms. Their structure is determined from averaging over "fast" spatial oscillations. The first term of the expansion with respect to the nonlinearity is $-T|A|^2A$. Therefore, in this approximation the equation

$$\partial A/\partial t = \gamma_0 A + \alpha \partial^2 A/\partial z^2 - T|A|^2 A \tag{1.2}$$

for the amplitudes has almost the same structure as the analogous equation of [1], differing from it only by the diffusion terms. It should be noted that this equation is encountered in the description of one-dimensional, weakly supercritical convection [7].

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The expansion terms not allowed for in (1.2) will be small only with the condition of positivity of the matrix element T. The matrix element T can be obtained from the Navier–Stokes equation by the standard scheme: It arises from perturbation theory in the third order with respect to $\sqrt{\varepsilon}$. We do not need the explicit form of T, however, Only information about the sign of T is required.

Let us consider the simplest solution of Eq. (1.2),

$$A_0 = (\gamma_0/T)^{1/2},$$

describing a mild mode of excitation at $T>0$. The existence of a solution of this type was proven in [8]. For just this reason $T>0$.

2. Equation (1.2) also has other steady-state solutions,

$$A = \mathrm{const}\, e^{iqz},\ |A|^2 = \frac{\gamma_0 - \alpha q^2}{T} \equiv \frac{\gamma_{k_0+q}}{T}, \tag{2.1}$$

which describe Taylor vortices with a period $2\pi/(k_0+q)$ differing from the optimum period $2\pi/k_0$. It is seen that such solutions exist only in that interval of wave numbers where the original flow is unstable ($\gamma_k>0$).

Within the framework of the equation for the envelopes we can analyze the stability of Taylor vortices with a given k. By linearizing Eq. (1.2) against the background of the steady-state equation (2.1), for perturbations of the form $\delta A = Ae^{\sigma t}u$ we can obtain the equation

$$\sigma u = \alpha \frac{\partial^2 u}{\partial z^2} + 2i\alpha q \frac{\partial u}{\partial z} - \gamma_{k_0+q}(u+u^*).$$

Its solution

$$u = u_1 e^{i\varkappa z} + u_2 e^{-i\varkappa z}$$

describes two branches of oscillations with decrements

$$\sigma = -\varkappa^2\alpha - \gamma_{k_0+q} \pm \left(\gamma_{k_0+q}^2 + 4\alpha^2 q^2\varkappa^2\right)^{1/2}.$$

From this it follows that Taylor vortices with a given q are stable [9] when

$$q^2 < \frac{1}{3}\frac{\gamma_0}{\alpha} \equiv \frac{1}{3}(\Delta k)^2.$$

We note that at the stability limit the amplitude of vortices with a given q remains finite and comprises $\sqrt{2/3}$ of the amplitude A (at the same supercriticality) of vortices with the optimum period $2\pi/k_0$. Therefore, with a decrease in the supercriticality ε, a system of vortices must reorganize, forming vortices with a period close to the optimum one. Also the reorganization must take place smoothly in an infinite system and with a jump in a finite one owing to the discreteness of k.

Now let us consider the general steady-state solution of Eq. (1.2). First we change to the dimensionless variables

$$\tau = \gamma_0 t,\ \xi = z/\sqrt{\alpha},\ u = A\sqrt{T} = x + iy,$$

in which the equation acquires the form

$$\partial u/\partial\tau = \partial^2 u/\partial\xi^2 + (1 - |u|^2)u. \tag{2.2}$$

Its steady-state solutions are determined from the equation

$$\partial^2 \mathbf{r}/\partial\xi^2 = (r^2 - 1)\mathbf{r},$$

which describes the "motion" of a particle in a centrally symmetric field with a potential

$$V = -(1 - r^2)^2/4.$$

This equation conserves the momentum $M = r^2\partial\theta/\partial\xi$ and the energy

$$E = \frac{1}{2}\left(\frac{\partial r}{\partial\xi}\right)^2 + \frac{M^2}{2r^2} - \frac{1}{4}(1-r^2)^2.$$

Finite motion, which exists only when $M^2 < 4/27$, has physical meaning. In this region of the parameters the roots e_1, e_2, and e_3 of the cubic equation $E = \frac{M^2}{2e} - \frac{1}{4}(1-e)^2$ are positive ($e_3>e_2>e_1>0$).

Then the solution of the equation is written in the form

$$r^2 = e_1 + (e_2 - e_1)\,\mathrm{sn}^2\nu\xi, \tag{2.3}$$

where $\nu = ((e_3-e_1)/2)^{1/2}$; $\mathrm{sn}\,\nu\xi$ is an elliptic sine with a modulus $k = ((e_2-e_1)/(e_3-e_1))^{1/2}$. In this case the phase

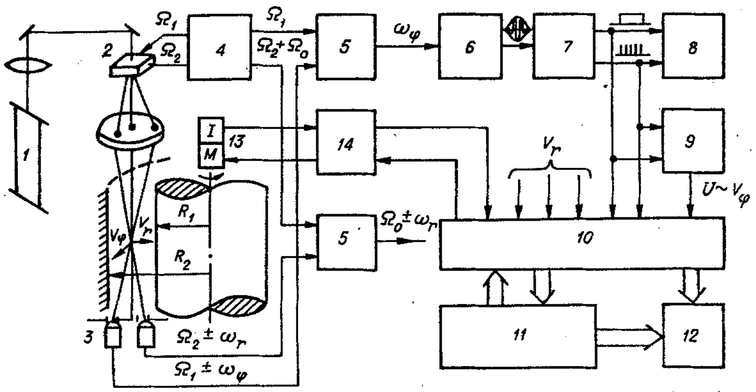

Fig. 1

θ is determined by integration of the expression M/r^2. Thus, the general steady-state solution found describes smooth modulations in the vortex system. We note that the Taylor vortices proper correspond to solutions with r^2 = const; the value of the constant is determined from the extrema of the effective potential energy $V + M^2/2r^2$. Here unstable Taylor vortices (2.1) ($q^2 > \Delta k^2/3$) correspond to a minimum while stable ones ($q^2 < \Delta k^2/3$) correspond to a maximum.

To investigate the stability of the general solution (2.3) we must determine the spectrum σ of the linearized equation (2.2),

$$\sigma\boldsymbol{\rho} = \partial^2\boldsymbol{\rho}/\partial\xi^2 + \boldsymbol{\rho}(1 - r^2) - 2\mathbf{r}(\boldsymbol{\rho}\mathbf{r}),$$

where $\boldsymbol{\rho}$ is the perturbation.

We represent the vector $\boldsymbol{\rho}$ in the form $\boldsymbol{\rho} = \varphi\mathbf{r} + \psi\partial\mathbf{r}/\partial\xi$. Here the functions φ and ψ are determined from the system

$$\begin{aligned}\sigma\varphi &= \partial^2\varphi/\partial\xi^2 - 2(1 - r^2)\partial\psi/\partial\xi - 2\varphi r^2,\\ \sigma\psi &= \partial^2\psi/\partial\xi^2 + 2\partial\varphi/\partial\xi,\end{aligned} \tag{2.4}$$

the coefficients of which are periodic functions of ξ with periods $2K/\nu$ ($K(k)$ is a complete elliptic integral of the first kind). Therefore, the solution (2.4) can be represented in the form of a product of $e^{ip\xi}$ times a periodic function of ξ, while the spectrum σ has a zonal structure as a function of the quasimomentum p. The boundaries of the zone corresponding to purely periodic solutions can be found explicitly.

Thus, two eigenfunctions $\boldsymbol{\rho}_1 = \partial\mathbf{r}/\partial\xi$ ($\psi = 1$, $\varphi = 0$) and $\boldsymbol{\rho}_2 = (y, -x)$ $\left(\varphi = -r\dfrac{\partial r}{\partial\xi}\Big/M, \quad \psi = r^2/M\right)$ having a quasimomentum $p = 0$ correspond to the value $\sigma = 0$. To find the other boundaries, we set $\psi = \partial f/\partial\xi$, in which case the system (2.4) is reduced to one fourth-order equation

$$f^{\mathrm{IV}} + 2f''(\sigma - 2 + 3r^2) + f\sigma(\sigma + 2r^2) = 0.$$

We present the periodic solutions of this equation and the values of σ corresponding to them, which represent the boundaries of the zones:

$$\begin{aligned}f &= \operatorname{sn}\nu\xi, \quad \sigma = (1/2)\{-(e_2 + e_3) \pm [(e_2 + e_3)^2 + 3(e_3 - e_2)^2]^{1/2}\},\\ f &= \operatorname{cn}\nu\xi, \quad \sigma = (1/2)\{-(e_1 + e_3) \pm [(e_1 + e_3)^2 + 3(e_3 - e_1)^2]^{1/2}\},\\ f &= \operatorname{dn}\nu\xi, \quad \sigma = (1/2)\{-(e_1 + e_2) \pm [(e_1 + e_2)^2 + 3(e_2 - e_1)^2]^{1/2}\}.\end{aligned}$$

For each of the solutions the value of the eigenvalue σ with the upper sign is positive, which corresponds to instability. Thus, in the weakly supercritical region there are no other stable steady-state solutions but Taylor vortices.

3. The experimental installation (Fig. 1) consists of a hydrodynamic stand with a precision drive, a laser Doppler velocity meter (LDVM), computer data input and output units, and external data output devices.

The hydrodynamic part of the installation consists of two coaxial metal cylinders 300 mm high, the gap between which is 10 mm when the inner cylinder is 35 mm in diameter. The radial wobble of the inner cylinder does not exceed 5 μm. A plastic outer cylinder of the same size was used in order to visualize the flow. Water or an aqueous solution of glycerin was used as the working liquid. A given liquid temperature was maintained with an accuracy of 0.02°C. A system of automatic control 14 of the speed of the motor 13 turning the inner cylinder provides stability on the order of 0.01% in a range from 100 msec to 10 sec. The rotation period can be varied with a minimum step of 10^{-4}.

To measure the liquid velocity we used a two-frequency, two-component, laser Doppler velocity meter the optical part of which is constructed on the scheme of forward scattering with a reference beam. As the device for shifting the emission frequency of the laser 1 and a splitter we used a Bragg acoustooptical cell 2 based on a TeO_2 crystal with an excitation frequency $\Omega_1 = 24$ MHz assigned by a frequency synthesizer 4. As shown in Fig. 1, the laser beams enter through the upper optical window into the gap between the cylinders. The geometry of the system allowed us to measure the velocity component V_φ. The size of the measurement volume was $600 \times 100 \times 100$ μm.

The use of a Bragg acoustooptical cell permitted a cardinal solution of the problem of filtration of the "pedestal" of the Doppler signal. The zero-order beam and the diffracted beams are focused in the test region of the stream, forming two interference fields with mutually orthogonal bands. The output signal of the photoreceiver 3 is sent to the input of a selective amplifier with a central tuning frequency of 24 MHz. This amplifier filters the noise and the constant component from the photoreceiver signal. Then the signal is fed to a mixer 5, where its spectrum is shifted to the region of Doppler frequencies, after which it is fed to a tunable filter 6, where the final filtration of noise and parasitic products of the frequency conversion from it occurs. The filter output is connected to a shaper 7 [10, 11], at the output of which one obtains packets of pulses with a Doppler frequency ω_φ which correspond to the passage of scattering particles through the measurement volume and gate pulses whose duration is equal to the duration of these packets. The signals from the shaper output are sent to a pulse counter and a time-interval meter, respectively, in a CAMAC block 10 and then to the memory of an M-400 computer 11, where a mass of values of the instantaneous velocity is stored. The information obtained is the initial information for the calculation of the required statistical characteristics of the velocity. The results obtained are printed out on a graph plotter 12. Operational monitoring is performed from an average-velocity indicator 8 and the signal of a frequency discriminator 9, the output voltage of which is proportional to the "instantaneous" velocity. The electronic units of the LDVM, the input–output unit, and the motor-drive control unit are built on the CAMAC standard [12]. The measurement complex also allows one to obtain information on the radial velocity component V_r. An auxiliary frequency shift Ω_0 is used to determine the sign of this component. The channel for amplification and treatment of the Doppler frequency ω_r is analogous to that described above.

4. The transition from the laminar state to Taylor vortices was studied in the experiments. It was established visually that different numbers N of pairs of vortices (from 11 to 18) are formed in the system beyond the stability threshold, depending on the initial conditions. With a metal outer cylinder in the installation the possibility of scanning along the z axis within the limits of two vortices also permitted a determination of the number N. It was found that with a decrease in Re to the critical Re_* a sequence of transitions was observed from a state with an arbitrary number N to a state with $N_0 = 14$. The dependence of the frequency $F = \Omega/2\pi$ of rotation of the inner cylinder at which a loss of stability of a system with a certain N occurs on the number N of vortex pairs is shown in Fig. 2. It is seen that the experimental transition points lie on a curve close to a parabola, which is in accordance with the theoretical concepts of Sec. 1.

The Landau law was tested for a system with the optimum N_0. For this we determined the amplitude A of the first spatial harmonic A along z in the interval of supercriticalities $0.01 < \varepsilon < 0.5$; the experimental dependence $A \sim \varepsilon^\gamma$, where $\gamma = 0.50 \pm 0.01$, confirms the Landau law with good accuracy. For comparison we point out that in analogous experiments [13] $\gamma = 0.50 \pm 0.03$.

It should be noted that at a slight supercriticality the small reserve of stability leads to the appearance of a high noise level. Therefore, only the data (Fig. 3) at supercriticalities $\varepsilon \approx 1\%$ are reliable. At the same

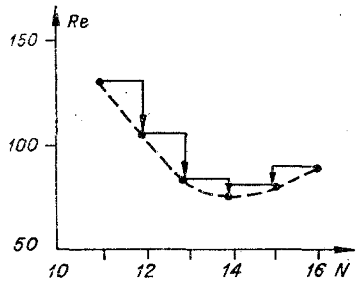

Fig. 2

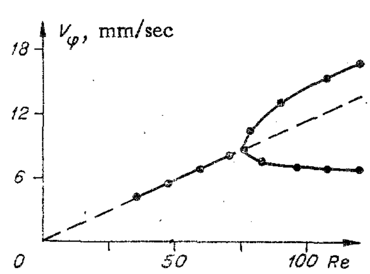

Fig. 3

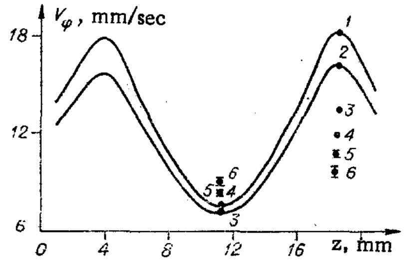

Fig. 4

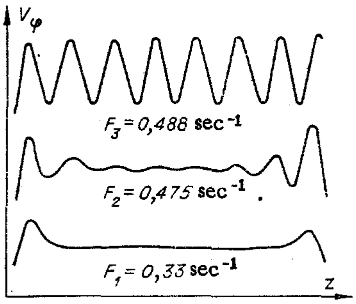

Fig. 5

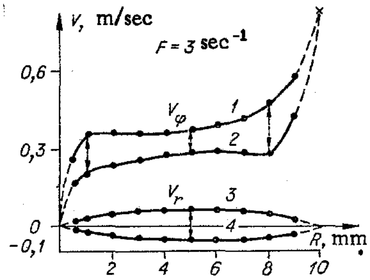

Fig. 6

time, data in the region of $\varepsilon \approx 0.1\%$ are contained in the experiments of [13]. Moreover, with a decrease in the supercriticality from $\varepsilon = 0.8$ to $\varepsilon = 0.018$ there is a shift of the extremal points 1-6 along z for the V_φ component, leading to disruption of the periodicity of the vortex structure (Fig. 4). It was shown experimentally that this is connected with the influence of the ends, near which the hydrodynamic noise level is high. The process of formation of vortices along the entire length was studied in one of the variants of the hydrodynamic stand with cylinders of smaller length (100 mm) and a 10-mm gap. Dependences of V_φ on z for three values of F are presented in Fig. 5, from which it is seen that vortices are created near the ends and gradually fill the entire system uniformly. With the same cylinders we obtained distributions of the components V_φ (curves 1 and 2 in Fig. 6) and V_r (curves 3 and 4) as a function of r for Taylor vortices at points corresponding to the maximum and minimum along z. It is seen that V_r is greatest at the center of the gap, while V_φ varies more complexly, forming two maxima. With an increase in supercriticality Taylor vortices become unstable relative to bending oscillations, studied visually in detail in [5]. A statistical analysis carried out on a computer shows that in this case the autocorrelation function describes weakly damped oscillations with a frequency ω_0 and a quality of $\sim 10^2$. With an increase in the supercriticality the form of the autocorrelation function becomes more complicated; frequencies not commensurable with ω_0 appear.

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STOKES FLOWS INSIDE A SPHERE

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1. Pair of Stream Functions of Three-Dimensional Flow

Let Ω be a sphere of radius R with the center at the origin of coordinates, $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\varphi$ be unit vectors of the spherical coordinate system (r, θ, φ), and $\mathbf{r} = r\mathbf{e}_r$. We designate the space of infinitely differentiable solenoidal vector fields in the closed sphere $\bar{\Omega}$ as V. In V we isolate the subspaces

$$V^- = \{\mathbf{v} \in V | \mathbf{v}\cdot\mathbf{e}_r = 0\},\ V^+ = \{\mathbf{v} \in V | \operatorname{rot}\mathbf{v}\cdot\mathbf{e}_r = 0\}.$$

With any function $F \in C^\infty(\bar{\Omega})$ one can associate a field $\mathbf{v}^-(F) = \operatorname{rot} F\mathbf{r} \in V^-$. Conversely, if a field $\mathbf{v}^- \in V^-$ is given, then from the condition $\operatorname{div}\mathbf{v}^- = 0$ we get

$$\frac{\partial}{\partial\theta}\left(v_\theta^- \sin\theta\right) + \frac{\partial v_\varphi^-}{\partial\varphi} = 0,$$

and therefore there exists a function $F^- \in C^\infty(\bar{\Omega})$ such that

$$\frac{\partial F^-}{\partial\varphi} = v_\theta^- \sin\theta,\quad \frac{\partial F^-}{\partial\theta} = -v_\varphi^-.$$

It is verified that $\mathbf{v}^- = \mathbf{v}^-(F^-) = \operatorname{rot} F^-\mathbf{r}$.

With any function $F \in C^\infty(\bar{\Omega})$ one can also associate a field $\mathbf{v}^+(F) = \operatorname{rot}\operatorname{rot} F\mathbf{r} \in V^+$. If a field $\mathbf{v}^+ \in V^+$ is given, then from the condition $\operatorname{rot}\mathbf{v}^+\cdot\mathbf{e}_r = 0$ we get

$$\frac{\partial}{\partial\theta}\left(v_\varphi^+ \sin\theta\right) - \frac{\partial v_\theta^+}{\partial\varphi} = 0,$$

and therefore a function $G \in C^\infty(\bar{\Omega})$ exists such that

$$\frac{\partial G}{\partial\varphi} = v_\varphi^+ \sin\theta,\quad \frac{\partial G}{\partial\theta} = v_\theta^+.$$

Defining $F^+(r, \theta, \varphi) = \frac{1}{r}\int\limits_0^r \rho G(\rho, \theta, \varphi)\,d\rho$, we can verify that the angular field components $\operatorname{rot}\operatorname{rot} F^+\mathbf{r}$ coincide with the corresponding components $\mathbf{v}^+$, and since the radial component of the field $\mathbf{v}$ without a singularity at the origin of coordinates is uniquely expressed through the angular components from the condition $\operatorname{div}\mathbf{v} = 0$, we have $\mathbf{v}^+ = \mathbf{v}^+(F^+) = \operatorname{rot}\operatorname{rot} F^+\mathbf{r}$.

The correspondences $F \to \mathbf{v}^-(F)$ and $F \to \mathbf{v}^+(F)$ defined above agree with the taking of the rot and lead to a scalar Laplace operator $\Delta = \operatorname{div}\operatorname{grad}$: $C^\infty(\bar{\Omega}) \to C^\infty(\bar{\Omega})$ and a vector operator $\Delta = -\operatorname{rot}\operatorname{rot}$: $V \to V$. That is, the following equations are valid: $\operatorname{rot}\mathbf{v}^-(F) = \mathbf{v}^+(F)$, $\operatorname{rot}\mathbf{v}^+(F) = -\mathbf{v}^-(\Delta F)$, $\Delta\mathbf{v}^\pm(F) = \mathbf{v}^\pm(\Delta F)$. Since $\mathbf{v}^-(F) = \operatorname{rot} F\mathbf{r} = \operatorname{grad} F \times \mathbf{r}$, the streamlines of the field $\mathbf{v}^-(F)$ are intersections of surfaces F = const with spheres r = const; F is the stream function for $\mathbf{v}^-(F)$. If the field $\mathbf{v}^+(F)$ is irrotational, then $\frac{\partial}{\partial r}(rF)$ is its potential. If $\mathbf{v}^+(F)$ has axial symmetry, then its stream function has the form $\Psi = -r\sin\theta\,\partial F/\partial\theta$. Since all potential fields and all axisymmetric fields without twist and with the condition of solenoidality belong to V^+, F generalizes the stream function and the potential of the field $\mathbf{v}^+(F)$ at the same time.

We can show that any field $\mathbf{v} \in V$ is represented uniquely in the form $\mathbf{v} = \mathbf{v}^- + \mathbf{v}^+$, where $\mathbf{v}^- \in V^-$ and $\mathbf{v}^+ \in V^+$. We determine the function $F^+(r, \theta, \varphi)$ for each fixed r, $0 < r \le R$, as the solution of the equation $\Delta_{\theta\varphi} F = -rv_r$ satisfying the condition

$$\iint\limits_{S_r} F^+ dS = 0, \tag{1.1}$$

where S_r is a sphere of radius r concentric with Ω and

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